# Markscheme

## May 2022

## Mathematics: analysis and approaches

## Higher level

## Paper 1

31 pages

## Instructions to Examiners

### Abbreviations

***M*** Marks awarded for attempting to use a correct **Method**.

***A*** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

***R*** Marks awarded for clear **Reasoning**.

***AG*** Answer given in the question and so no marks are awarded.

***FT*** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

### Using the markscheme

**1 General**

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2.***

**2 Method and Answer/Accuracy marks**

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

| | **Correct answer seen** | **Further working seen** | **Any FT issues?** | **Action** |
|---|---|---|---|---|
| **1.** | $8\sqrt{2}$ | $5.65685...$ *(incorrect decimal value)* | No. Last part in question. | Award ***A1*** for the final mark *(condone the incorrect further working)* |
| **2.** | $\frac{35}{72}$ | $0.468111\ldots$ (incorrect decimal value) | Yes. Value is used in subsequent parts. | Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)* |

## 3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

## 4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

**For example**: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.

- If the candidate's answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.

- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was "Hence".

## 5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.

- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than $1$, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- Miscopying of candidates' own work does **not** constitute a misread, it is an error.

- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.

- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

## 6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

## 7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example $1.9$ and $1,9$ or $1000$ and $1,000$ and $1.000$.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

## 8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

**Simplification of final answers:** Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as $2$, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4\mathrm{e}^{2x}\times\mathrm{e}^{3x}$ should be simplified to $4\mathrm{e}^{5x}$, and $4\mathrm{e}^{2x}\times\mathrm{e}^{3x}-\mathrm{e}^{4x}\times\mathrm{e}^{x}$ should be simplified to $3\mathrm{e}^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2+x$ are both acceptable.

**Please note:** intermediate ***A*** marks do NOT need to be simplified.

## 9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

## 10. Presentation of candidate work

**Crossed out work:** If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

**More than one solution:** Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

## Section A

**1.** (a) $u_1 = 12$ ***A1***

***[1 mark]***

(b) $15 - 3n = -33$ ***(A1)***

$n = 16$ ***A1***

***[2 marks]***

(c) valid approach to find $d$ ***(M1)***

$u_2 - u_1 = 9 - 12$ OR recognize gradient is $-3$ OR attempts to solve $-33 = 12 + 15d$

$d = -3$ ***A1***

***[2 marks]***

***Total [5 marks]***

**2.** (a) $(n-1)+n+(n+1)$ ***(A1)***

$=3n$ ***A1***

which is always divisible by $3$ ***AG***

***[2 marks]***

(b) $(n-1)^2+n^2+(n+1)^2$ $(=n^2-2n+1+n^2+n^2+2n+1)$ ***A1***

attempts to expand either $(n-1)^2$ or $(n+1)^2$ (do not accept $n^2-1$ or $n^2+1$) ***(M1)***

$=3n^2+2$ ***A1***

demonstrating recognition that 2 is not divisible by 3 or $\frac{2}{3}$ seen after correct expression divided by 3 ***R1***

$3n^2$ is divisible by $3$ and so $3n^2+2$ is never divisible by $3$

OR the first term is divisible by 3, the second is not

OR $3\left(n^2+\frac{2}{3}\right)$ OR $\frac{3n^2+2}{3}=n^2+\frac{2}{3}$

hence the sum of the squares is never divisible by $3$ ***AG***

***[4 marks]***

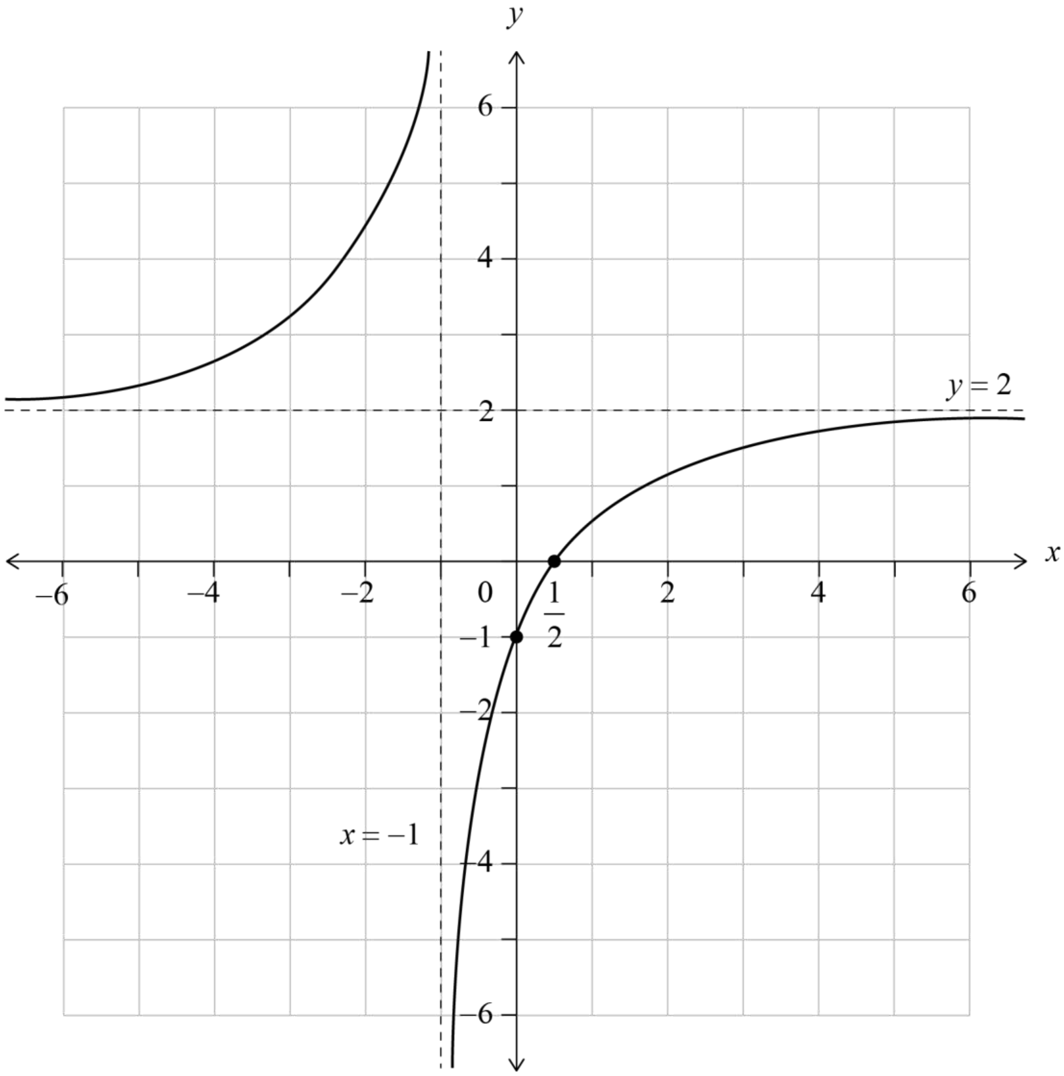
***Total [6 marks]***

**3.** (a) (i) $x = -1$ ***A1***

(ii) $y = 2$ ***A1***

***[2 marks]***

(b)



rational function shape with two branches in opposite quadrants, with two correctly positioned asymptotes and asymptotic behaviour shown ***A1***

axes intercepts clearly shown at $x=\frac{1}{2}$ and $y=-1$ ***A1A1***

***[3 marks]***

*continued…*

*Question 3 continued*

(c) $x > \frac{1}{2}$ ***A1***

| |
|---|
| **Note:** Accept correct alternative correct notation, such as $\left(\frac{1}{2}, \infty\right)$ and $\left]\frac{1}{2}, \infty\right[$. |

***[1 mark]***

(d) **EITHER**

attempts to sketch $y = \frac{2\|x\|-1}{\|x\|+1}$ ***(M1)***

**OR**

attempts to solve $2\|x\|-1=0$ ***(M1)***

| |
|---|
| **Note:** Award the ***(M1)*** if $x = \frac{1}{2}$ and $x = -\frac{1}{2}$ are identified. |

**THEN**

$x < -\frac{1}{2}$ or $x > \frac{1}{2}$ ***A1***

| |
|---|
| **Note:** Accept the use of a comma. Condone the use of ‘and’. Accept correct alternative notation. |

***[2 marks]***

***Total [8 marks]***

**4.** determines $\frac{\pi}{4}$ (or $45°$) as the first quadrant (reference) angle ***(A1)***

attempts to solve $\frac{x}{2}+\frac{\pi}{3}=\frac{\pi}{4}$ ***(M1)***

**Note:** Award ***M1*** for attempting to solve $\frac{x}{2}+\frac{\pi}{3}=\frac{\pi}{4},\frac{7\pi}{4}(,...)$

$\frac{x}{2}+\frac{\pi}{3}=\frac{\pi}{4}\Rightarrow x<0$ and so $\frac{\pi}{4}$ is rejected ***(R1)***

$\frac{x}{2}+\frac{\pi}{3}=2\pi-\frac{\pi}{4}\ \left(=\frac{7\pi}{4}\right)$ ***A1***

$x=\frac{17\pi}{6}$ (must be in radians) ***A1***

***[5 marks]***

**5.** (a) **EITHER**

recognises the required term (or coefficient) in the expansion ***(M1)***

$bx^5 = {}^7C_2\, x^5 1^2$ OR $b = {}^7C_2$ OR ${}^7C_5$

$b = \frac{7!}{2!5!} \left( = \frac{7!}{2!(7-2)!} \right)$

correct working ***A1***

$\frac{7\times6\times5\times4\times3\times2\times1}{2\times1\times5\times4\times3\times2\times1}$ OR $\frac{7\times6}{2!}$ OR $\frac{42}{2}$

**OR**

lists terms from row $7$ of Pascal's triangle ***(M1)***

$1, 7, 21, ...$ ***A1***

**THEN**

$b = 21$ ***AG***

***[2 marks]***

(b) $a = 7$ ***(A1)***

correct equation ***A1***

$21x^5 = \frac{ax^6 + 35x^4}{2}$ OR $21x^5 = \frac{7x^6 + 35x^4}{2}$

correct quadratic equation ***A1***

$7x^2 - 42x + 35 = 0$ OR $x^2 - 6x + 5 = 0$ (or equivalent)

valid attempt to solve **their** quadratic ***(M1)***

$(x-1)(x-5) = 0$ OR $x = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(5)}}{2(1)}$

$x = 1,\ x = 5$ ***A1***

**Note:** Award final ***A0*** for obtaining $x = 0,\ x = 1,\ x = 5$.

***[5 marks]***

***Total [7 marks]***

**6.** (a) attempts to replace $x$ with $-x$ ***M1***

$f(-x) = -x\sqrt{1-(-x)^2}$

$= -x\sqrt{1-(-x)^2}\ (= -f(x))$ ***A1***

| |
|---|
| **Note:** Award ***M1A1*** for an attempt to calculate both $f(-x)$ and $-f(-x)$ independently, showing that they are equal. |
| **Note:** Award ***M1A0*** for a graphical approach including evidence that **either** the graph is invariant after rotation by $180°$ about the origin **or** the graph is invariant after a reflection in the $y$- axis and then in the $x$- axis (or vice versa). |

so $f$ is an odd function ***AG***

***[2 marks]***

(b) attempts both product rule and chain rule differentiation to find $f'(x)$ ***M1***

$f'(x) = x \times \frac{1}{2} \times (-2x) \times (1-x^2)^{-\frac{1}{2}} + (1-x^2)^{\frac{1}{2}} \times 1 \quad \left(= \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}}\right)$ ***A1***

$= \frac{1-2x^2}{\sqrt{1-x^2}}$

sets their $f'(x) = 0$ ***M1***

$\Rightarrow x = \pm\frac{1}{\sqrt{2}}$ ***A1***

attempts to find at least one of $f\left(\pm\frac{1}{\sqrt{2}}\right)$ ***(M1)***

| |
|---|
| **Note:** Award ***M1*** for an attempt to evaluate $f(x)$ at least at one of their $f'(x) = 0$ roots. |

$a = -\frac{1}{2}$ and $b = \frac{1}{2}$ ***A1***

| |
|---|
| **Note:** Award ***A1*** for $-\frac{1}{2} \le y \le \frac{1}{2}$. |

***[6 marks]***

***Total [8 marks]***

**7. METHOD 1**

$u = \sec x \Rightarrow \mathrm{d}u = \sec x \tan x \, \mathrm{d}x$ ***(A1)***

attempts to express the integral in terms of $u$ ***M1***

$\int_1^2 u^{n-1} \mathrm{d}u$ ***A1***

$= \frac{1}{n}\left[u^n\right]_1^2 \quad \left(= \frac{1}{n}\left[\sec^n x\right]_0^{\frac{\pi}{3}}\right)$ ***A1***

| **Note:** Condone the absence of or incorrect limits up to this point. |
|---|

$= \frac{2^n - 1^n}{n}$ ***M1***

$= \frac{2^n - 1}{n}$ ***A1***

| **Note:** Award ***M1*** for correct substitution of **<u>their</u>** limits for $u$ into their antiderivative for $u$ ( or given limits for $x$ into their antiderivative for $x$). |
|---|

**METHOD 2**

$\int \sec^n x \tan x \, \mathrm{d}x = \int \sec^{n-1} x \sec x \tan x \, \mathrm{d}x$ ***(A1)***

applies integration by inspection ***(M1)***

$= \frac{1}{n}\left[\sec^n x\right]_0^{\frac{\pi}{3}}$ ***A2***

| **Note:** Award ***A2*** if the limits are not stated. |
|---|

$= \frac{1}{n}\left(\sec^n \frac{\pi}{3} - \sec^n 0\right)$ ***M1***

| **Note:** Award ***M1*** for correct substitution into their antiderivative. |
|---|

$= \frac{2^n - 1}{n}$ ***A1***

***[6 marks]***

**8.** let $m$ be the median

**EITHER**

attempts to find the area of the required triangle **M1**

base is $(m-a)$ **(A1)**

and height is $\dfrac{2}{(b-a)(c-a)}(m-a)$

area $=\dfrac{1}{2}(m-a)\times\dfrac{2}{(b-a)(c-a)}(m-a)\ \left(=\dfrac{(m-a)^2}{(b-a)(c-a)}\right)$ **A1**

**OR**

attempts to integrate the correct function **M1**

$$\int_a^m \frac{2}{(b-a)(c-a)}(x-a)\,\mathrm{d}x$$

$=\dfrac{2}{(b-a)(c-a)}\left[\dfrac{1}{2}(x-a)^2\right]_a^m$ OR $\dfrac{2}{(b-a)(c-a)}\left[\dfrac{x^2}{2}-ax\right]_a^m$ **A1A1**

| |
|---|
| **Note:** Award ***A1*** for correct integration and ***A1*** for correct limits. |

**THEN**

sets up (their) $\displaystyle\int_a^m \frac{2}{(b-a)(c-a)}(x-a)\,\mathrm{d}x$ or area $=\dfrac{1}{2}$ **M1**

| |
|---|
| **Note:** Award ***M0A0A0M1A0A0*** if candidates conclude that $m>c$ and set up their area or sum of integrals $=\dfrac{1}{2}$. |

$$\frac{(m-a)^2}{(b-a)(c-a)}=\frac{1}{2}$$

$m=a\pm\sqrt{\dfrac{(b-a)(c-a)}{2}}$ **(A1)**

as $m>a$, rejects $m=a-\sqrt{\dfrac{(b-a)(c-a)}{2}}$

so $m=a+\sqrt{\dfrac{(b-a)(c-a)}{2}}$ **A1**

***[6 marks]***

**9.** **METHOD 1 (rearranging the equation)**

assume there exists some $\alpha \in \mathbb{Z}$ such that $2\alpha^3 + 6\alpha + 1 = 0$ ***M1***

| |
|---|
| **Note:** Award ***M1*** for equivalent statements such as 'assume that $\alpha$ is an integer root of $2\alpha^3 + 6\alpha + 1 = 0$'. Condone the use of $x$ throughout the proof. Award ***M1*** for an assumption involving $\alpha^3 + 3\alpha + \frac{1}{2} = 0$. |
| **Note:** Award ***M0*** for statements such as "let's consider the equation has integer roots…" ,"let $\alpha \in \mathbb{Z}$ be a root of $2\alpha^3 + 6\alpha + 1 = 0$ …" |
| **Note:** Subsequent marks after this ***M1*** are independent of this ***M1*** and can be awarded. |

attempts to rearrange their equation into a suitable form ***M1***

**EITHER**

$2\alpha^3 + 6\alpha = -1$ ***A1***

$\alpha \in \mathbb{Z} \Rightarrow 2\alpha^3 + 6\alpha$ is even ***R1***

$2\alpha^3 + 6\alpha = -1$ which is not even and so $\alpha$ cannot be an integer ***R1***

| |
|---|
| **Note:** Accept '$2\alpha^3 + 6\alpha = -1$ which gives a contradiction'. |

**OR**

$1 = 2\left(-\alpha^3 - 3\alpha\right)$ ***A1***

$\alpha \in \mathbb{Z} \Rightarrow \left(-\alpha^3 - 3\alpha\right) \in \mathbb{Z}$ ***R1***

$\Rightarrow 1$ is even which is not true and so $\alpha$ cannot be an integer ***R1***

| |
|---|
| **Note:** Accept '$\Rightarrow 1$ is even which gives a contradiction'. |

*continued…*

*Question 9 continued*

**OR**

$\frac{1}{2} = -\alpha^3 - 3\alpha$ **A1**

$\alpha \in \mathbb{Z} \Rightarrow \left(-\alpha^3 - 3\alpha\right) \in \mathbb{Z}$ **R1**

$-\alpha^3 - 3\alpha$ is not an integer $\left(= \frac{1}{2}\right)$ and so $\alpha$ cannot be an integer **R1**

| |
|---|
| **Note:** Accept '$-\alpha^3 - 3\alpha$ is not an integer $\left(= \frac{1}{2}\right)$ which gives a contradiction'. |

**OR**

$\alpha = -\frac{1}{2\left(\alpha^2 + 3\right)}$ **A1**

$\alpha \in \mathbb{Z} \Rightarrow -\frac{1}{2\left(\alpha^2 + 3\right)} \in \mathbb{Z}$ **R1**

$-\frac{1}{2\left(\alpha^2 + 3\right)}$ is not an integer and so $\alpha$ cannot be an integer **R1**

| |
|---|
| **Note:** Accept $-\frac{1}{2\left(\alpha^2 + 3\right)}$ is not an integer which gives a contradiction'. |

**THEN**

so the equation $2x^3 + 6x + 1 = 0$ has no integer roots **AG**

***[5 marks]***

*continued…*

*Question 9 continued*

**METHOD 2**

assume there exists some $\alpha \in \mathbb{Z}$ such that $2\alpha^3 + 6\alpha + 1 = 0$ ***M1***

**Note:** Award ***M1*** for statements such as 'assume that $\alpha$ is an integer root of $2\alpha^3 + 6\alpha + 1 = 0$'. Condone the use of $x$ throughout the proof. Award ***M1*** for an assumption involving $\alpha^3 + 3\alpha + \frac{1}{2} = 0$ and award subsequent marks based on this.

**Note:** Award ***M0*** for statements such as "let's consider the equation has integer roots…", "let $\alpha \in \mathbb{Z}$ be a root of $2\alpha^3 + 6\alpha + 1 = 0$…"

**Note:** Subsequent marks after this ***M1*** are independent of this ***M1*** and can be awarded.

let $f(x) = 2x^3 + 6x + 1$ (and $f(\alpha) = 0$)

$f'(x) = 6x^2 + 6 > 0$ for all $x \in \mathbb{R}$ $\Rightarrow f$ is a (strictly) increasing function ***M1A1***

$f(0) = 1$ and $f(-1) = -7$ ***R1***

thus $f(x) = 0$ has only one real root between $-1$ and $0$, which gives a contradiction

(or therefore, contradicting the assumption that $f(\alpha) = 0$ for some $\alpha \in \mathbb{Z}$), ***R1***

so the equation $2x^3 + 6x + 1 = 0$ has no integer roots ***AG***

***[5 marks]***

## Section B

**10.** (a) uses $\sum \mathrm{P}(X = x) = 1$ to form a linear equation in $p$ and $q$ ***(M1)***

correct equation in terms of $p$ and $q$ from summing to $1$ ***A1***

$p + 0.3 + q + 0.1 = 1$ OR $p + q = 0.6$ (or equivalent)

uses $\mathrm{E}(X) = 2$ to form a linear equation in $p$ and $q$ ***(M1)***

correct equation in terms of $p$ and $q$ from $\mathrm{E}(X) = 2$ ***A1***

$p + 0.6 + 3q + 0.4 = 2$ OR $p + 3q = 1$ (or equivalent)

| **Note:** The marks for using $\sum \mathrm{P}(X = x) = 1$ and the marks for using $\mathrm{E}(X) = 2$ may be awarded independently of each other. |
|---|

evidence of correctly solving these equations simultaneously ***A1***

for example, $2q = 0.4 \Rightarrow q = 0.2$ or $p + 3 \times (0.6 - p) = 1 \Rightarrow p = 0.4$

so $p = 0.4$ and $q = 0.2$ ***AG***

***[5 marks]***

(b) valid approach ***(M1)***

$\mathrm{P}(X > 2) = \mathrm{P}(X = 3) + \mathrm{P}(X = 4)$ OR $\mathrm{P}(X > 2) = 1 - \mathrm{P}(X = 1) - \mathrm{P}(X = 2)$

$= 0.3$ ***A1***

***[2 marks]***

*continued…*

*Question 10 continued*

(c) recognises at least one of the valid scores (6, 7, or 8) required to win the game ***(M1)***

| **Note:** Award ***M0*** if candidate also considers scores other than 6, 7, or 8 (such as 5). |
|---|

let $T$ represent the score on the last two rolls

a score of $6$ is obtained by rolling $(2,4),(4,2)$ or $(3,3)$

$\mathrm{P}(T=6)=2(0.3)(0.1)+(0.2)^2 \quad (=0.1)$ ***A1***

a score of $7$ is obtained by rolling $(3,4)$ or $(4,3)$

$\mathrm{P}(T=7)=2(0.2)(0.1) \quad (=0.04)$ ***A1***

a score of $8$ is obtained by rolling $(4,4)$

$\mathrm{P}(T=8)=(0.1)^2 \quad (=0.01)$ ***A1***

| **Note:** The above 3 ***A1*** marks are independent of each other. |
|---|

$\mathrm{P}(\text{Nicky wins})=0.1+0.04+0.01$

$=0.15$ ***A1***

***[5 marks]***

(d) $3+b=8$ ***(M1)***

$b=5$ ***A1***

***[2 marks]***

*continued…*

*Question 10 continued*

(e) **METHOD 1**

**EITHER**

$\mathrm{P}(S=5)=\frac{4}{16}$

$\mathrm{P}(S=a+2)=\frac{4}{16}$ ***A1***

$\Rightarrow a+2=5$

**OR**

$\mathrm{P}(S=6)=\frac{3}{16}$

$\mathrm{P}(S=a+3)=\frac{2}{16}$ and $\mathrm{P}(S=5+1)=\frac{1}{16}$ ***A1***

$\Rightarrow a+3=6$

**OR**

$\mathrm{P}(S=4)=\frac{3}{16}$

$\mathrm{P}(S=a+1)=\frac{2}{16}$ and $\mathrm{P}(S=1+3)=\frac{1}{16}$ ***A1***

$\Rightarrow a+1=4$

**THEN**

$\Rightarrow a=3$ ***A1***

| **Note:** Award ***A0A0*** for $a=3$ obtained without working/reasoning/justification. |
|---|

***[2 marks]***

*continued…*

*Question 10 continued*

**METHOD 2**

**EITHER**

correctly lists a relevant part of the sample space ***A1***

for example, $\{S=4\}=\{(3,1),(1,a),(1,a)\}$ or $\{S=5\}=\{(2,a),(2,a),(2,a),(2,a)\}$

or $\{S=6\}=\{(3,a),(3,a),(1,5)\}$

$a+3=6$

**OR**

eliminates possibilities (exhaustion) for $a<5$

convincingly shows that $a\neq 2,4$ ***A1***

$a\neq 4$, for example, $\mathrm{P}(S=7)=\frac{2}{16}$ from $(2,5),(2,5)$ and so $(3,a),(3,a)\Rightarrow a+3\neq 7$

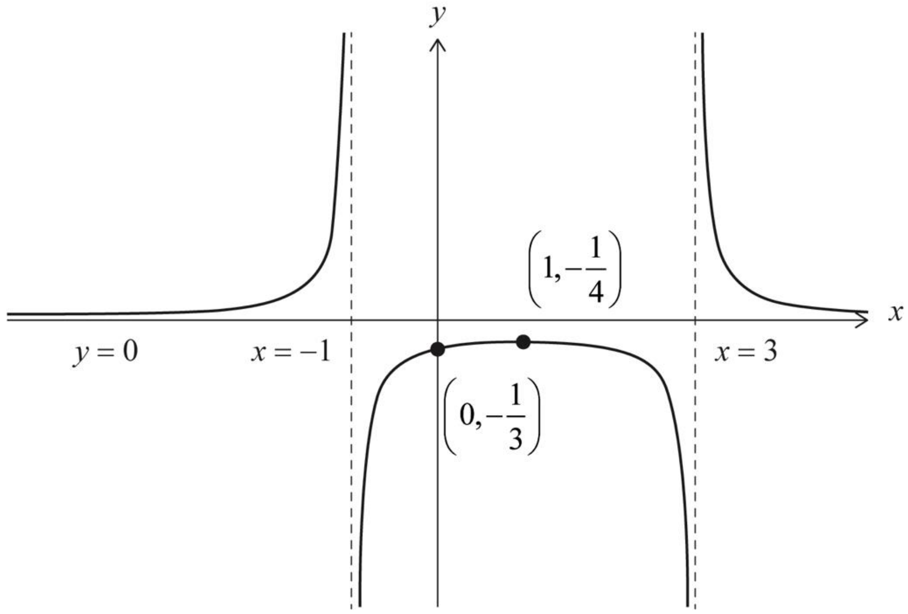
**THEN**

$\Rightarrow a=3$ ***A1***

***[2 marks]***

***Total [16 marks]***

**11.** (a)




$y$- intercept $\left(0,-\frac{1}{3}\right)$ ***A1***

| |
|---|
| **Note:** Accept an indication of $-\frac{1}{3}$ on the $y-$axis. |

vertical asymptotes $x=-1$ and $x=3$ ***A1***

horizontal asymptote $y=0$ ***A1***

uses a valid method to find the $x$- coordinate of the local maximum point ***(M1)***

| |
|---|
| **Note:** For example, uses the axis of symmetry or attempts to solve $f'(x)=0$. |

local maximum point $\left(1,-\frac{1}{4}\right)$ ***A1***

| |
|---|
| **Note:** Award ***(M1)A0*** for a local maximum point at $x=1$ and coordinates not given. |

three correct branches with correct asymptotic behaviour and the key features in approximately correct relative positions to each other ***A1***

***[6 marks]***

*continued…*

*Question 11 continued*

(b) (i) $x = \dfrac{1}{y^2 - 2y - 3}$ ***M1***

| **Note:** Award ***M1*** for interchanging $x$ and $y$ (this can be done at a later stage). |
|---|

**EITHER**

attempts to complete the square ***M1***

$y^2 - 2y - 3 = (y-1)^2 - 4$ ***A1***

$x = \dfrac{1}{(y-1)^2 - 4}$

$(y-1)^2 - 4 = \dfrac{1}{x}$ $\left((y-1)^2 = 4 + \dfrac{1}{x}\right)$ ***A1***

$y - 1 = \pm\sqrt{4 + \dfrac{1}{x}}$ $\left(= \pm\sqrt{\dfrac{4x+1}{x}}\right)$

**OR**

attempts to solve $xy^2 - 2xy - 3x - 1 = 0$ for $y$ ***M1***

$y = \dfrac{-(-2x) \pm \sqrt{(-2x)^2 + 4x(3x+1)}}{2x}$ ***A1***

| **Note:** Award ***A1*** even if – (in $\pm$) is missing |
|---|

$= \dfrac{2x \pm \sqrt{16x^2 + 4x}}{2x}$ ***A1***

*continued…*

*Question 11 continued*

**THEN**

$=1\pm\dfrac{\sqrt{4x^2+x}}{x}$ ***A1***

$y>3$ and hence $y=1-\dfrac{\sqrt{4x^2+x}}{x}$ is rejected ***R1***

**Note:** Award ***R1*** for concluding that the expression for $y$ must have the '$+$' sign. The ***R1*** may be awarded earlier for using the condition $x>3$.

$y=1+\dfrac{\sqrt{4x^2+x}}{x}$

$g^{-1}(x)=1+\dfrac{\sqrt{4x^2+x}}{x}$ ***AG***

(ii) domain of $g^{-1}$ is $x>0$ ***A1***

***[7 marks]***

*continued…*

*Question 11 continued*

(c) attempts to find $(h \circ g)(a)$ ***(M1)***

$(h \circ g)(a) = \arctan\left(\frac{g(a)}{2}\right)$ $\left((h \circ g)(a) = \arctan\left(\frac{1}{2(a^2 - 2a - 3)}\right)\right)$ ***(A1)***

$\arctan\left(\frac{g(a)}{2}\right) = \frac{\pi}{4}$ $\left(\arctan\left(\frac{1}{2(a^2 - 2a - 3)}\right) = \frac{\pi}{4}\right)$

attempts to solve for $g(a)$ ***M1***

$\Rightarrow g(a) = 2$ $\left(\frac{1}{(a^2 - 2a - 3)} = 2\right)$

**EITHER**

$\Rightarrow a = g^{-1}(2)$ ***A1***

attempts to find their $g^{-1}(2)$ ***M1***

$a = 1 + \frac{\sqrt{4(2)^2 + 2}}{2}$ ***A1***

**Note:** Award all available marks to this stage if $x$ is used instead of $a$.

**OR**

$\Rightarrow 2a^2 - 4a - 7 = 0$ ***A1***

attempts to solve their quadratic equation ***M1***

$a = \frac{-(-4) \pm \sqrt{(-4)^2 + 4(2)(7)}}{4}$ $\left(= \frac{4 \pm \sqrt{72}}{4}\right)$ ***A1***

**Note:** Award all available marks to this stage if $x$ is used instead of $a$.

**THEN**

$a = 1 + \frac{3}{2}\sqrt{2}$ (as $a > 3$) ***A1***

$(p = 1,\ q = 3,\ r = 2)$

**Note:** Award ***A1*** for $a = 1 + \frac{1}{2}\sqrt{18}$ $(p = 1,\ q = 1,\ r = 18)$.

***[7 marks]***

***Total [20 marks]***

**12.** (a) $z_2^* = r_2 e^{-i\theta}$ ***(A1)***

$z_1 z_2^* = r_1 e^{i\alpha} r_2 e^{-i\theta}$ ***A1***

$z_1 z_2^* = r_1 r_2 e^{i(\alpha-\theta)}$ ***AG***

| **Note:** Accept working in modulus-argument form |
|---|

***[2 marks]***

(b) $\operatorname{Re}\left(z_1 z_2^*\right) = r_1 r_2 \cos(\alpha-\theta) \quad (=0)$ ***A1***

$\alpha - \theta = \arccos 0 \ \left(r_1, r_2 > 0\right)$

$\alpha - \theta = \frac{\pi}{2}$ (as $0 < \alpha - \theta < \pi$) ***A1***

so $Z_1OZ_2$ is a right-angled triangle ***AG***

***[2 marks]***

(c) (i) **EITHER**

$\frac{z_1}{z_2}\left(= \frac{r_1}{r_2} e^{i(\alpha-\theta)}\right) = e^{i\frac{\pi}{3}}$ (since $r_1 = r_2$) ***(M1)***

**OR**

$z_1 = r_2 e^{i\left(\theta+\frac{\pi}{3}\right)} \left(= r_2 e^{i\theta} e^{i\frac{\pi}{3}}\right)$ ***(M1)***

**THEN**

$z_1 = z_2 e^{i\frac{\pi}{3}}$ ***A1***

| **Note:** Accept working in either modulus-argument form to obtain $z_1 = z_2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ or in Cartesian form to obtain $z_1 = z_2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$. |
|---|

*continued…*

*Question 12 continued*

(ii) substitutes $z_1 = z_2 e^{i\frac{\pi}{3}}$ into $z_1^2 + z_2^2$ ***M1***

$z_1^2 + z_2^2 = z_2^2 e^{i\frac{2\pi}{3}} + z_2^2 \left(= z_2^2\left(e^{i\frac{2\pi}{3}} + 1\right)\right)$ ***A1***

**EITHER**

$e^{i\frac{2\pi}{3}} + 1 = e^{i\frac{\pi}{3}}$ ***A1***

**OR**

$z_2^2\left(e^{i\frac{2\pi}{3}} + 1\right) = z_2^2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i + 1\right)$

$= z_2^2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$ ***A1***

**THEN**

$z_1^2 + z_2^2 = z_2^2 e^{i\frac{\pi}{3}}$

$= z_2\left(z_2 e^{i\frac{\pi}{3}}\right)$ and $z_2 e^{i\frac{\pi}{3}} = z_1$ ***A1***

so $z_1^2 + z_2^2 = z_1 z_2$ ***AG***

| |
|---|
| **Note:** For candidates who work on the LHS and RHS separately to show equality, award ***M1A1*** for $z_1^2 + z_2^2 = z_2^2 e^{i\frac{2\pi}{3}} + z_2^2 \left(= z_2^2\left(e^{i\frac{2\pi}{3}} + 1\right)\right)$, ***A1*** for $z_1 z_2 = z_2^2 e^{i\frac{\pi}{3}}$ and ***A1*** for $e^{i\frac{2\pi}{3}} + 1 = e^{i\frac{\pi}{3}}$. Accept working in either modulus-argument form or in Cartesian form. |

***[6 marks]***

*continued…*

*Question 12 continued*

(d) **METHOD 1**

$z_1 + z_2 = -a$ and $z_1 z_2 = b$ ***(A1)***

$a^2 = z_1{}^2 + z_2{}^2 + 2z_1 z_2$ ***A1***

$a^2 = 2z_1 z_2 + z_1 z_2 \left(= 3z_1 z_2\right)$ ***A1***

substitutes $b = z_1 z_2$ into their expression ***M1***

$a^2 = 2b + b$ OR $a^2 = 3b$ ***A1***

| **Note:** If $z_1 + z_2 = -a$ is not clearly recognized, award maximum ***(A0)A1A1M1A0.*** |
|---|

so $a^2 - 3b = 0$ ***AG***

**METHOD 2**

$z_1 + z_2 = -a$ and $z_1 z_2 = b$ ***(A1)***

$\left(z_1 + z_2\right)^2 = z_1{}^2 + z_2{}^2 + 2z_1 z_2$ ***A1***

$\left(z_1 + z_2\right)^2 = 2z_1 z_2 + z_1 z_2 \left(= 3z_1 z_2\right)$ ***A1***

substitutes $b = z_1 z_2$ and $z_1 + z_2 = -a$ into their expression ***M1***

$a^2 = 2b + b$ OR $a^2 = 3b$ ***A1***

| **Note:** If $z_1 + z_2 = -a$ is not clearly recognized, award maximum ***(A0)A1A1M1A0.*** |
|---|

so $a^2 - 3b = 0$ ***AG***

***[5 marks]***

(e) $a^2 - 3 \times 12 = 0$

$a = \pm 6 \ \left(\Rightarrow z^2 \pm 6z + 12 = 0\right)$ ***A1***

for $a = -6$:

$z_1 = 3 + \sqrt{3}\mathrm{i},\ z_2 = 3 - \sqrt{3}\mathrm{i}$ and $\alpha - \theta = -\frac{5\pi}{3}$ which does not satisfy $0 < \alpha - \theta < \pi$ ***R1***

for $a = 6$:

$z_1 = -3 - \sqrt{3}\mathrm{i},\ z_2 = -3 + \sqrt{3}\mathrm{i}$ and $\alpha - \theta = \frac{\pi}{3}$ ***A1***

so (for $0 < \alpha - \theta < \pi$), only one equilateral triangle can be formed from point O and the two roots of this equation ***AG***

***[3 marks]***

***Total [18 marks]***